Fractional-Stokes limit for kinetic equations

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Classical Stokes hydrodynamical limit

Boltzmann equation:

$$\partial_t h(t,x,v) + v \cdot \nabla_x h(t,x,v) \quad = \quad Q(h,h), \qquad h = M + \delta g(t,x,v)$$

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Theorem ([Golse and Levermore, 2002])

$$g_\varepsilon \rightharpoonup g = \left(\underbrace{\int g\, dv}_{=:\,\rho} + \underbrace{\int vg\, dv}_{=:\,\vec{m}} \cdot v + \underbrace{\int \left(\frac{1}{d}\|v\|^2 - 1\right) g\, dv}_{=:\,\theta} \left(\frac{1}{2}\|v\|^2 - \frac{d}{2}\right)\right) M$$

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$$\textit{where}\quad \begin{cases}\nabla_x\cdot\vec{m} = 0, \qquad \rho + \theta = 0,\\ \partial_t\vec{m} = \omega\Delta_x\vec{m} + \nabla_x p \quad + \textit{init.},\\ \dfrac{d+2}{2}\partial_t\theta = \kappa\Delta_x\theta \quad + \textit{init.}\end{cases}$$

Our problem

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Theorem (?)

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Why bother?

Anomalous transport phenomena.

fractional Laplacian

Heavy-tailed distribution functions.

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- Interest? Anomalous transport phenomena.
- Meaning of the fractional Laplacian and relation to the classical one.
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$$F(v) \sim \|v\|^{-(d+\alpha)} \quad \alpha := \sup_{a>0} \left\{ \int_{\mathbb{R}^d} \|v\|^a F(v) dv < \infty \right\}$$

Anomalous Transport Phenomena

- Diffusion equation

$$\partial_t \rho(t,x) = D\Delta_x \rho(t,x)$$

Mean Square Displacement:

- Super-diffusion (variance $\uparrow \infty$)

fractional Fick's law $\Rightarrow$ fractional diffusion equation

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$$\begin{cases} \partial_t \rho(t,x) &= -\nabla_x \cdot \vec{\jmath}(t,x) \quad \text{(mass cons.)} \\ \vec{\jmath}(t,x) &= -D\nabla_x \rho(t,x) \quad \text{(Fick's law)} \end{cases} \Rightarrow \partial_t \rho(t,x) = D\Delta_x \rho(t,x)$$

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  - Change to a fractional Fick's law $\Rightarrow$ fractional diffusion equation

$$\vec{\jmath} = -D\nabla_x^{\alpha - 1}\rho \quad \alpha \in (1,2) \implies \partial_t \rho = D\Delta_x^{\alpha/2}\rho$$

Fractional Laplacian

- Fourier-Transform

$$(-\Delta_x)^{\alpha/2}\rho := \mathcal{F}^{-1}\left(|k|^{\alpha}\mathcal{F}(\rho)(k)\right)$$

- Principal Value Integral

$$(-\Delta_x)^{\alpha/2}\rho := c_{d,\alpha}\int_{\mathbb{R}^d}\frac{\rho(x)-\rho(x+y)}{\|y\|^{d+\alpha}}dy$$

Fractional diffusion limit: models conserving the 0th moment

Theorem

Micro.
(atomic view)

Hydrodyn.
limits

Macro.
Observable

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$$\partial_t f + v \cdot \nabla_x f = L(f)$$

$$L(f) = \left(\int f dv\right) M(v) - f$$

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Rescaling

$t \mapsto \varepsilon^2 t$ *(speed time)*

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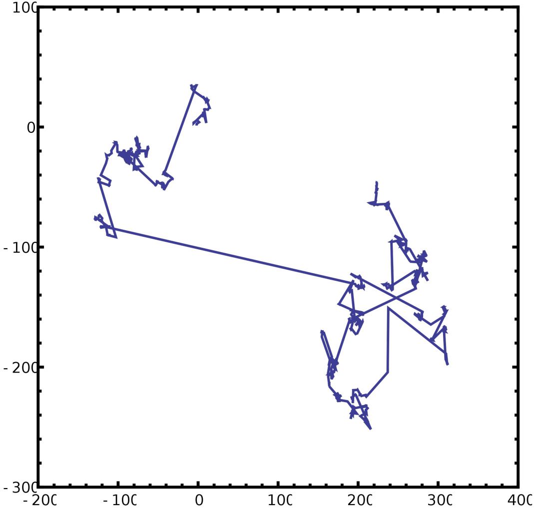
- *Fourier-Laplace transform [Mellet, Mischler, and Mouhot, 2011]*
- *Moment's method [Mellet, 2010], [Ben Abdallah, Mellet, and Puel, 2011]*
- *Fractional Hilbert expansion [Abdallah, Mellet, and Puel, 2010]*

Why faster? Stochastic interpretation([1])

- Markov Process



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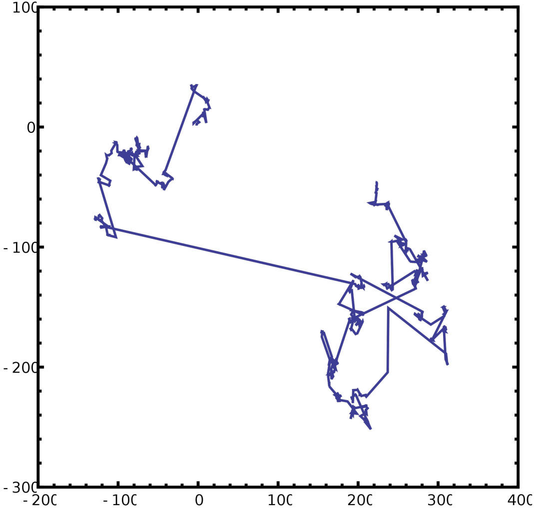

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  - Brownian motion

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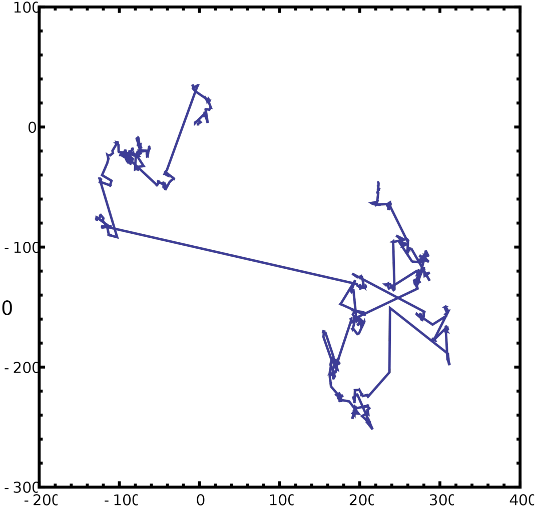

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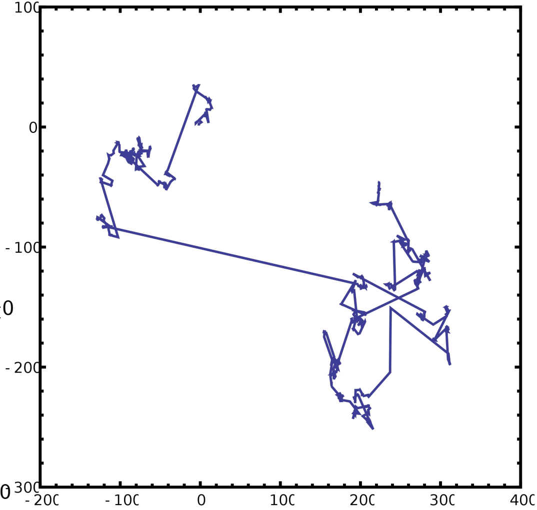

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where

$$\kappa = \int L^{-1}(v) \otimes v \, M dv \approx - \int \frac{\|v\|^2}{\nu} M dv$$

Our goal

L, γ?

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Problem (?) We need finite second moment to define the temperature.

Construction of the linear kinetic equation

$$\partial_t f + v \cdot \nabla_x f = \nu K(f) - \nu f, \qquad \nu(v) \sim \|v\|^{\beta},\ \beta \in \mathbb{R}$$

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$$\int_{\mathbb{R}^d} F(v)\, dv = 1, \quad \int_{\mathbb{R}^2} \|v\|^2 F(v)\, dv = d, \quad \int_{\mathbb{R}^d} \|v\|^4 F(v)\, dv = d(d+2)$$

$$\tilde{F}(v) \sim \|v\|^{-(d+\alpha)}, \qquad F^*(v) = (2\pi)^{d/2} \exp(-\|v\|^2/2)$$

Theorem (Fractional Stokes-Fourier (Hittmeir, Merino, 2014))

Suppose $f|_{t=0} = f^{in} \in L^2(F^{-1}dv)$ *and*

- **Heavy-tail case:** $\alpha > 5$ *and* $\beta < 1$ *with*

  $$5 < \alpha + \beta < 6, \qquad \beta < \frac{\alpha - 4}{2}, \qquad \gamma = \frac{\alpha - \beta - 4}{1 - \beta} \in (1,2);$$

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then the Cauchy problem is well defined; and

$$f_\varepsilon \rightharpoonup \left(\rho + (v \cdot \vec{m}) + \frac{(\|v\|^2 - d)}{2}\theta \right) F \quad L^2_{\nu F^{-1}}((0,T); L^2(\mathbb{R}^{2d})) - weak$$

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$$f_\varepsilon \rightharpoonup \left(\rho + (v \cdot \vec{m}) + \frac{(\|v\|^2 - d)}{2}\theta \right) F \quad L^2_{\nu F^{-1}}((0, T); L^2(\mathbb{R}^{2d})) - weak$$

where, in the sense of distributions,

$$\begin{aligned} \partial_t \theta &= -C(-\Delta_x)^{\gamma/2}\theta \qquad \theta(0, x) = \theta^{in} \\ \vec{m}(t, x) &= \vec{m}^{in} \end{aligned}$$

Theorem (Fractional Stokes-Fourier (Hittmeir, Merino, 2014))

Suppose $f|_{t=0} = f^{in} \in L^2(F^{-1}dv)$ *and*

- **Heavy-tail case:** $\alpha > 5$ *and* $\beta < 1$ *with*

$$5 < \alpha + \beta < 6, \qquad \beta < \frac{\alpha - 4}{2}, \qquad \gamma = \frac{\alpha - \beta - 4}{1 - \beta} \in (1, 2);$$

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Theorem (Fractional Stokes (Hittmeir, Merino, 2014))

Suppose $f|_{t=0} = f^{in} \in L^2(F^{-1}dv)$ *and a* **heavy-tail equilibrium***:* $\alpha > 3$ *and* $\beta < 1$ *with*

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Some comments...

Strategy: moments method (Antoine Mellet)

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- Explicit solution

$$\chi_\varepsilon(t,x,v) = \int_0^\infty e^{-\nu z} \nu\, \varphi(t, x + \varepsilon v z)\, dz$$

- Convergence results for χ_ε

Weak formulation

$$\varepsilon^\gamma \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon = \nu K(f_\varepsilon) - \nu f_\varepsilon$$

Multiply the equation by $\psi(v) = 1, v, (\|v\|^2 - d)/2$ and χ_ε and integrate (by parts).

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& \left(\text{and since } \int_{\mathbb{R}^N} \psi\nu f\, dv = \int \psi\nu K(f)\, dv\right) \\
= & \ \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2d}} \nu\psi K(f_\varepsilon)(\chi_\varepsilon - \varphi)\, dvdxdt
\end{aligned}$$

Computing the fractional symbol

$$\varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2d}} \nu \|v\|^4 \theta_{\varepsilon,\nu} M(\chi_\varepsilon - \varphi)\, dvdxdt$$

$=$

$=$

$\asymp$

$$\asymp \quad \varepsilon^{-\gamma+(\alpha-\beta-4)/(1-\beta)} \int_{\|w\|\geq\varepsilon} \int_0^\infty e^{-s} \frac{\varphi(t,x+ws)-\varphi(t,x)}{\|w\|^{N+(\alpha-\beta-4)/(1-\beta)}}\, dsdw$$

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Large velocities:

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$$= \varepsilon^{-\gamma} \int_{\|v\|\geq 1} \nu \|v\|^4 M \int_0^\infty e^{-\nu z} \nu \left(\varphi(t, x + \varepsilon v z) - \varphi(t,x)\right) dz\, dv$$

$$=$$

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\asymp\ & \varepsilon^{-\gamma} \int_{\|v\|\geq 1} \|v\|^{4+\beta-N-\alpha} \int_0^\infty e^{-s} \left(\varphi(t, x + \underbrace{\varepsilon v/\|v\|^\beta}_{=w} s) - \varphi(t,x) \right) dsa \\
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